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Limit Distribution For Semi-Markovian Random Walk With A Generalized Delaying Barrier

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ABSTRACT

In this study, a semi – Markovian random walk process ($X(t)$) with a generalized delaying barrieris considered andergodic theorem for this process is proved under some weak conditions. Moreover, the characteristic function of the ergodic distribution of the process $X(t)$ is expressed by characteristic function of a boundary functional $S_{N(z)}$. Then, using this representation, it is shown that the ergodic distribution of the "standardized" process $Y_\lambda(t) \equiv X(t)/\lambda$ converges to a limit distribution, when $\lambda \to \infty$. Finally, the explicit form of the limit distribution is obtained.
Key words: *Semi - Markovian random walk; Delaying barrier; Ergodic distribution; Asymptotic expansion; Weak convergence; Limit distribution.*

INTRODUCTION

Many interesting problems of stochastic finance, mathematical biology, reliability, queuing, stock control theories and mathematical insurance can be expressed by means of random walk processes. In particular, a number of very interesting problems of stock control, queuing and reliability theories are expressed by means of random walk with various types of barriers. These barriers can be reflecting, delaying, absorbing, elastic, etc., depending on concrete problems at hand. For instance, it is possible to express random levels of stock in a warehouse with finite (or infinite) volumes or queuing systems with finite (or infinite) waiting time or sojourn time by means of random walk with delaying barriers (or barrier). Furthermore, the functioning of stochastic systems with spare equipment can be given by random walk with barriers, one of them is delaying and the other one is any barrier. In this topic, there are many interesting studies in literature [1, 4, 7, 12].

Unfortunately, the results of these studies are not readily applicable to real – world problems because the probability characteristics of considered processes have very complex mathematical structure. For avoiding this difficulty, in recent years, the asymptotic methods for investigation of the processes of queuing, reliability, stock control, etc. theories are intensively developed. In these topics, there are also some important researches in literature [1, 2, 3, 4, 7, 9, 10].

However, these studies have only dealt with the boundary functionals of random walks. Doubtlessly, the boundary functionals of stochastic processes are extremely important. But own characteristics of the random walk are also important for solving various applied problems. For this reason, we are interested in ergodic distribution of the semi – Markovian random walk ($X(t)$) with a generalized delayingbarrier. Namely, in this study we will investigate the asymptotic behavior of the ergodic distribution of the process $X(t)$.Due to a special property of the barrier, we expect that the ergodic distribution of the process $X(t)$converges to a limit distribution of a "residual waiting time", which is important for real – world applications.

These kinds of problems may occur, for example, in the control of military stocks, refinery stocks, reserve of oil wells, stochastic finance, mathematical insurance, etc.

MATERIALS AND METHODS

Let $\{(\xi_n, \eta_n)\}$, n=1,2,...be a sequence of independent and identically distributed random pairs defined on a probability space $(\Omega, \Im, P)$, where ξ_n take only positive values, whereas η_n take both positive and negative values. Suppose that the random variables of ξ_n and η_n are mutually independent and theirdistribution functions are given as follows:

$$\Phi(t) = P\{\xi_n \leq t\}, \; F(x) = P\{\eta_n \leq x\}.$$

Define renewal sequence $\{T_n\}$ and random walk $\{S_n\}$ as follows:

$$T_n = \sum_{i=1}^{n} \xi_i, \quad S_n = \sum_{i=1}^{n} \eta_i, \qquad T_0 = S_0 = 0, \qquad n = 1,2,3 \dots$$

Moreover, let the sequence of random variables N_n, S_{N_n} and ζ_n bedefined as follows:

$$N_0 = 0; \; S_{N_0} = 0; \; \zeta_0 = z, N_1 \equiv N_1(\lambda z) \equiv N(\lambda z) = \inf\{n \geq 1 : \lambda z - S_n < 0\};$$

$$S_{N(\lambda z)} = \sum_{i=1}^{N(\lambda z)} \eta_i; \quad \zeta_1 = \left(-\eta_{N_1+L_1}\right)^+;$$

$$N_n = \inf\{r \geq 1 : \lambda\zeta_{n-1} - \left(S_{N_1+L_1+\dots+N_{n-1}+L_{n-1}+r} - S_{N_1+L_1\dots N_{n-1}+L_{n-1}}\right) < 0\};$$

$$S_{N_n} = \sum_{i=1}^{N_n} \eta_i; \; \zeta_n = (-\eta_{N_1+L_1+\dots N_n+L_n})^+, n \geq 1,$$

Here, $(-\eta_n)^+$ indicates positive part of the random variable $(-\eta_n)$, and the random variables L_1, L_2,...are number of jumping required to go up random walk S_n from zero to positive positions $\zeta_1, \zeta_2, \zeta_3, \dots$.

Let

$$\theta_1 = \sum_{i=1}^{L_1} \xi_{N_1+i}, \qquad \theta_n = \sum_{i=1}^{L_n} \xi_{N_1+L_1+\dots+N_{n-1}+L_{n-1}+N_n+i}, \qquad n = 1,2,3,\dots$$

and $\tau_n = T_{N_n}, \gamma_n = \tau_n + \theta_n, n = 1,2,3,\dots, \gamma_0 = \tau_0 = 0$.

Moreover,define $\nu(t) = \max\{n \geq 0 : T_n \leq t\}, t > 0$.

Now we can construct the desired stochastic process $X(t)$ as in the following form:

$$X(t) = \max\left\{0; \; \lambda\zeta_n - \left(S_{\nu(t)} - S_{N_1+L_1+\dots N_n+L_n}\right)\right\}, t \in [\gamma_n, \gamma_{n+1}), n = 0,1,2,\dots$$

where $\lambda > 0$.

One of trajectories of the process $X(t)$ is given as in Figure 1:

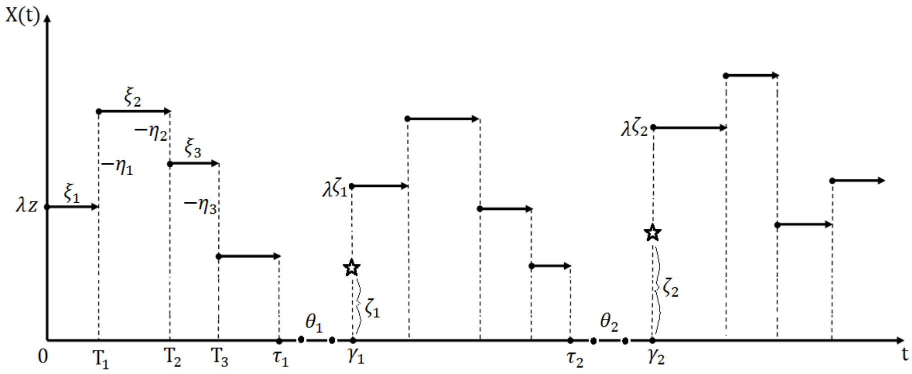

Figure 1: One of trajectories of the process X(t)

Note that, $\zeta_0 = z > 0$ and $\zeta_1, \zeta_2, \zeta_3, \dots$are independent and identically distributed positive valued random variableswithdistribution $\pi(z)$:

$$\pi(z) \equiv P\{\zeta_n \leq z\} = \frac{F(0) - F(-z)}{F(0)},$$

$$F(z) = P\{\eta_n \leq z\}, z \geq 0, n = 1,2,\dots$$

The process $X(t)$ is called "Semi – Markovian Random Walk Processwith a GeneralizedDelaying Barrier" and the main purpose of this study is to investigate the asymptotic behavior of the ergodic distribution of theprocess $X(t)$, when $\lambda \to \infty$.

RESULTS

ERGODICITY OF THE PROCESS $X(t)$

Firstly, we state the following theorem on the ergodicity of the process $X(t)$

Theorem 3.1. Let the initial sequence of random pairs $\{(\xi_n,\eta_n), n \geq 1\}$ be satisfied the following supplementary conditions:

$1)\ E(\xi_1) < \infty$; $2) E(\eta_1) > 0$; $3) P\{\eta_1 < 0\} > 0, P\{\eta_1 > 0\} > 0$;

4) η_1 is non-arithmetic random variable.

Then, the process $X(t)$ is ergodic and for any bounded measurable function $f:[0,\infty) \to R$, the following relation holds, with probability 1:

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f(X(s))ds = \frac{1}{E(\gamma_1)}\int_{z=0}^{\infty}\int_{x=0}^{\infty}\int_{t=0}^{\infty} f(x)\, P\{\gamma_1 > t; X(t) \in dx\}\, dt\, d\pi(z) \qquad (3.1)$$

Proof. The process $X(t)$ belongs to a wide class of processes which is called in literature as "The class of semi-Markov process with a discrete interference of chance". General ergodic theorem of type Smith`s "key renewal theorem" exists in the literature for this class (see, Gihman and Skorohod [7], p.243). It is not difficult to show that the assumptions of the general ergodic theorem are satisfied under the conditions of Theorem 3.1. Therefore, the ergodicity of the process $X(t)$ is derived from this general ergodic theorem.

Note. From Eq. (3.1) many valuable inferences can be derived. To state some of these inferences, let us introduce the following notations:

$$Q_X(x) \equiv \lim_{X\to\infty} P\{X(t) \leq x\};\quad \varphi_X(\alpha) = \lim_{t\to\infty} E(e^{i\alpha X(t)}), \alpha \in R.$$

Here, $Q_X(x)$ is ergodic distribution function and $\varphi_X(\alpha)$ is ergodic characteristic function of the process $X(t)$.

Corollary 3.1. The ergodic distribution function $(Q_X(x))$ of the process $X(t)$ has the following exact form:

$$Q_X(x) = \frac{1}{E(\gamma_1)}\int_{z=0}^{\infty}\int_{t=0}^{\infty} G(t,x,\lambda z)\, dt\, d\pi(z/\lambda),$$

where $G(t,x,\lambda z) = P_{\lambda z}\{\gamma_1 > t; X(t) \leq x\}$

Proof: Taking $f(x)$ as indicator function yields the desired result.

Corollary 3.2. The explicit form of characteristic function $(\varphi_X(\alpha))$ of ergodic distribution of the process $X(t)$ can be given as:

$$\varphi_X(\alpha) = \frac{1}{E(\gamma_1)}\int_{z=0}^{\infty}\int_{t=0}^{\infty}\int_{x=0}^{\infty} e^{i\alpha x} d_x\, G(t,x,\lambda z)\, dt\, d\pi(z/\lambda).$$

Proof: Taking $f(x) = \exp(i\alpha x)$ yields the desired result.

Using the basic identity for the random walk process (see, Feller [6], p.514), we can obtain an alternative representation for the characteristic function $(\varphi_X(\alpha))$ of the ergodic distribution.

Theorem 3.2. (Alternative representation for $\varphi_X(\alpha)$). Under the conditions of Theorem 3.1, the characteristic function $\varphi_X(\alpha)$ of the ergodic distribution of the process $X(t)$ can be expressed by means of characteristics of the boundary functional $S_{N(z)}$ as follows:

$$\varphi_X(\alpha) = \frac{1}{E(N_1(\lambda\zeta_1)) + K}\int_{z=0}^{\infty} e^{i\alpha\lambda z}\,\frac{\varphi_{S_{N(\lambda z)}}(-\alpha) - 1}{\varphi_\eta(-\alpha) - 1}\, d\pi(z) + \frac{K}{E(N_1(\lambda\zeta_1)) + K}\int_{z=0}^{\infty} e^{i\alpha\lambda z}\,\varphi_{S_{N(\lambda z)}}(-\alpha)\, d\pi(z),\quad \alpha \in R\backslash\{0\} \qquad (3.2)$$

where $\varphi_{S_{N(\lambda z)}}(-\alpha) \equiv E(\exp(-i\alpha S_{N(\lambda z)}))$, $\varphi_\eta(-\alpha) \equiv E(\exp(-i\alpha\eta_1))$,

$E(N_1(\lambda\zeta_1)) \equiv \int_0^\infty E(N_1(\lambda z))\, d\pi(z);\ K = \frac{1}{F(0)}$.

ASYMPTOTIC EXPANSIONS FOR THE MOMENTS OF $S_{N(z)}$

Firstly, give the following result related to the moments of the boundary functional $S_{N(z)}$ which exist in the literature (Khaniyev and Mammadova [9]).

Proposition 4.1. Assume that the following conditions are satisfied:

1) $E(\eta_1) > 0$, 2) $P\{\eta_1 < 0\} > 0, P\{\eta_1 > 0\} > 0$, 3) $E(|\eta_1|^3) < +\infty$,

4) η_1 is non-arithmetic random variable.

Then, the following asymptotic expansions can be written for the moments of $S_{N(z)}$, as $z \to \infty$:

$$M_n(z) \equiv E(S^n_{N(z)}) = z^n + n\mu_{21} z^{n-1} + o(z^{n-1}),\qquad n = \overline{1,3},$$

where $\mu_k \equiv E(\chi_1^{+k})$, $k \geq 1$; $\mu_{k1} = \frac{\mu_k}{k\mu_1}$, $k \geq 2$ and χ_1^+ is the first ascending ladder height of the random walk $\{S_n\}, n \geq 0$.

Proposition 4.2. Let g(x) bea bounded measurable function and $\lim_{x\to\infty} g(x) = 0$.Then the following asymptoticrelation holds, when $\lambda \to \infty$:

$$\int_0^\infty z^n g(\lambda z)\, d\pi(z) \to 0, \qquad n = 0, 1, 2, \ldots$$

Using the Proposition 4.1 and Proposition 4.2, we can state the following corollary.

Corollary4.1. Under the assumptions of Proposition 4.1 the following asymptotic expansions are true for the integrals from the moments of $S_{N(z)}$, as $\lambda \to \infty$:

$$E\left(\tilde{\zeta}_1^n M_1(\tilde{\zeta}_1)\right) = \lambda^{n+1}\beta_{n+1} + \lambda^n \mu_{11} \beta_n + o(\lambda^{n-1}), \qquad n = 0, 1, 2, \ldots$$

where $\tilde{\zeta}_1 \equiv \lambda \zeta_1$; $\beta_k \equiv E(\zeta_1^k)$, $k = 1, 2, \ldots, \zeta_1 = (-\eta_1)^+$ is positive part of random variable $(-\eta_1)$.

WEAK CONVERGENCETHEOREM FOR THE PROCESS $Y_\lambda(t)$

Let define a new process $Y_\lambda(t)$ as follows:

$$Y_\lambda(t) \equiv X(t)/\lambda, \ \lambda > 0.$$

In this section, weak convergence theorem for the distribution of the process $Y_\lambda(t)$ is proved and the limit form for the ergodic distribution of the process $Y_\lambda(t)$ is obtained, when $\lambda \to \infty$.Note that, since $Y_\lambda(t)$ is a linear transform of the process X(t),the process $Y_\lambda(t)$ is also ergodic.

Let denote the characteristic function of the ergodic distribution of the process $Y_\lambda(t)$ by $\varphi_Y(\alpha)$, that is,

$$\varphi_Y(\alpha) \equiv \lim_{t\to\infty} E\left(\exp\left(i\alpha Y_\lambda(t)\right)\right).$$

Before giving the weak convergence theorem, let usstate the following limit theorem for the characteristic function of the process $Y_\lambda(t)$.

Theorem 5.1. Under the conditions of Theorem 3.1, the characteristic function $(\varphi_Y(\alpha))$ of the ergodic distribution of the process $Y_\lambda(t)$converges to limitfunction $\varphi_0(\alpha)$,as $\lambda \to \infty$, that is,

$$\varphi_Y(\alpha) \to \varphi_0(\alpha) \equiv \frac{\varphi_\zeta(\alpha) - 1}{i\alpha E(\zeta_1)}, \ \alpha \neq 0.$$

Here $\varphi_\zeta(\alpha)$ is the characteristic function of the random variable ζ_1.

Proof. From Theorem 3.2, we have

$$\varphi_X(\alpha) = \frac{1}{E(N_1(\lambda\zeta_1)) + K} \int_{z=0}^{\infty} e^{i\alpha\lambda z} \frac{\varphi_{S_{N(\lambda z)}}(-\alpha) - 1}{\varphi_\eta(-\alpha) - 1} d\pi(z) + \frac{K}{E(N_1(\lambda\zeta_1)) + K} \int_{z=0}^{\infty} e^{i\alpha\lambda z} \varphi_{S_{N(\lambda z)}}(-\alpha) d\pi(z) \ , \quad \alpha \in R \backslash \{0\}. \tag{5.1}$$

Here

$$\varphi_{S_{N(\lambda z)}}(-\alpha) \equiv E\left(\exp(-i\alpha S_{N(\lambda z)})\right), \ \varphi_\eta(-\alpha) \equiv E(\exp(-i\alpha\eta_1)) \ ,$$

$$E\left(N_1(\lambda\zeta_1)\right) \equiv \int_0^\infty E\left(N_1(\lambda z)\right) d\pi(z).$$

On the other hand, the ergodic distribution of the process $Y_\lambda(t)$ is given as follows:

$$Q_Y(x) = \lim_{t\to\infty} P\{Y_\lambda(t) \le x\} = \lim_{t\to\infty} P\left\{\frac{X(t)}{\lambda} \le x\right\} = Q_X(\lambda x); \ \ x > 0. \tag{5.2}$$

Moreover,we can represent the characteristic function$(\varphi_Y(\alpha))$ of the ergodic distribution of the process $Y(t)$ as follows:

$$\varphi_Y(\alpha) = \lim_{t\to\infty} E\left(\exp\left(i\alpha Y_\lambda(t)\right)\right) = \lim_{t\to\infty} E\left(\exp\left(i\alpha \frac{X(t)}{\lambda}\right)\right) = \varphi_X\left(\frac{\alpha}{\lambda}\right). \tag{5.3}$$

So we get

$$\varphi_Y(\alpha) = \frac{1}{E(N_1(\tilde{\zeta}_1)) + K}\left\{\int_0^\infty e^{i\frac{\alpha}{\lambda}z} \frac{\varphi_{S_{N(z)}}\left(-\frac{\alpha}{\lambda}\right) - 1}{\varphi_\eta\left(-\frac{\alpha}{\lambda}\right) - 1} d\pi\left(\frac{z}{\lambda}\right) + K\int_0^\infty e^{i\frac{\alpha}{\lambda}z} \varphi_{S_{N(z)}}\left(-\frac{\alpha}{\lambda}\right) d\pi\left(\frac{z}{\lambda}\right)\right\}. \tag{5.4}$$

Let define the following notations:

$$R(\lambda) = \frac{1}{E(N_1(\zeta_1)) + K};\ I_1(\lambda) = \int_0^\infty e^{i\frac{\alpha}{\lambda}z} \frac{\varphi_{S_{N(z)}}\left(-\frac{\alpha}{\lambda}\right) - 1}{\varphi_\eta\left(-\frac{\alpha}{\lambda}\right) - 1} d\pi(z/\lambda);$$

$$I_2(\lambda) = \int_0^\infty \exp(i\frac{\alpha}{\lambda}z)\varphi_{S_{N(z)}}\left(-\frac{\alpha}{\lambda}\right) d\pi(z/\lambda).$$

Then, we have

$$\varphi_Y(\alpha) = R(\lambda)\{I_1(\lambda) + KI_2(\lambda)\}. \tag{5.5}$$

Thus, we get

$$I_1(\lambda) = \int_0^\infty \exp(i\alpha x) \frac{E(\exp(-i\frac{\alpha}{\lambda}S_{N(\lambda x)})) - 1}{E(\exp(-i\frac{\alpha}{\lambda}\eta_1)) - 1} d\pi(x),$$
$$= \int_0^\infty \frac{E\left\{\exp\left(-i\frac{\alpha}{\lambda}(S_{N(\lambda x)} - \lambda x)\right)\right\} - e^{i\alpha x}}{E(\exp(-i\frac{\alpha}{\lambda}\eta_1)) - 1} d\pi(x). \tag{5.6}$$

According to definition of $S_{N(\lambda x)}$, the following asymptotic expansion can be written:

$$E(S_{N(\lambda x)}) = \lambda x + \mu_{21} + o(1). \tag{5.7}$$

Taking account Eq. (5.7) into Eq. (5.6), it can be written:

$$I_1(\lambda) = \lambda\int_0^\infty \frac{e^{i\alpha x} - 1}{i\alpha m_1} d\pi(x) + o(1). \tag{5.8}$$

On the other hand,

$$R(\lambda) \equiv \frac{1}{E(N_1(\lambda\zeta_1)) + K} = \frac{m_1}{m_1 E(N_1(\lambda\zeta_1)) + Km_1} = \frac{m_1}{E(S_{N(\lambda\zeta_1)}) + Km_1}. \tag{5.9}$$

By using the asymptotic expansion in Corollary 4.1, we get the following expansion for $R(\lambda)$, as $\lambda \to \infty$:

$$R(\lambda) = \frac{m_1}{Km_1 + \lambda\beta_1 + \mu_{21} + o(1)}$$
$$= \frac{m_1}{\lambda\beta_1}\left\{1 - \frac{Km_1 + \mu_{21}}{\lambda\beta_1} + o(\frac{1}{\lambda})\right\} = \frac{m_1}{\lambda\beta_1}(1 + o(1)). \tag{5.10}$$

This yields to

$$R(\lambda)I_1(\lambda) = \frac{m_1}{\lambda\beta_1}\lambda\int_0^\infty \frac{e^{i\alpha x} - 1}{i\alpha m_1} d\pi(x)\{1 + o(1)\}$$
$$= \int_0^\infty \frac{e^{i\alpha x} - 1}{i\alpha\beta_1} d\pi(x) + o(1)$$
$$= \frac{E(e^{i\alpha\zeta_1}) - 1}{i\alpha E(\zeta_1)} + o(1) = \frac{\varphi_\zeta(\alpha) - 1}{i\alpha E(\zeta_1)} + o(1). \tag{5.11}$$

Here $\varphi_\zeta(\alpha) \equiv E(\exp(i\alpha\zeta_1)) \equiv \int_0^\infty \exp(i\alpha x)\, d\pi(x)$.

Therefore, we have the following limit relation:

$$\lim_{\lambda\to\infty} R(\lambda)I_1(\lambda) = \frac{\varphi_\zeta(\alpha) - 1}{i\alpha E(\zeta_1)}. \tag{5.12}$$

Now, let calculate $I_2(\lambda)$:

$$I_2(\lambda) = \int_0^\infty \exp(i\frac{\alpha}{\lambda}z)\varphi_{S_{N(z)}}\left(-\frac{\alpha}{\lambda}\right) d\pi(z/\lambda)$$
$$= \int_0^\infty E\left\{\exp(-i\frac{\alpha}{\lambda}(S_{N(\lambda x)} - \lambda x)\right\} d\pi(x)$$
$$= 1 - i\frac{\alpha}{\lambda}\int_0^\infty E\{S_{N(\lambda x)} - \lambda x\} d\pi(x) + o\left(\frac{1}{\lambda}\right). \tag{5.13}$$

From Corollary 4.1, the following asymptotic expansion can be obtained:

$$I_2(\lambda) = 1 - i\frac{\alpha}{\lambda}(\mu_{21} + o(1)) + o\left(\frac{1}{\lambda}\right) = 1 - \frac{i\alpha\mu_{21}}{\lambda} + o\left(\frac{1}{\lambda}\right). \tag{5.14}$$

From (5.10) and (5.14), we get the following asymptotic expansion, as $\lambda \to \infty$:

$$R(\lambda)KI_2(\lambda) = \frac{m_1}{\lambda\beta_1}K\left\{1 - \frac{i\alpha\mu_{21}}{\lambda} + o\left(\frac{1}{\lambda}\right)\right\} = \frac{Km_1}{\lambda\beta_1} + o\left(\frac{1}{\lambda}\right).$$

Since $K < \infty$, $m_1 < \infty$, $0 < \beta_1 < \infty$ and $\mu_{21} < \infty$, we get

$$\lim_{\lambda \to \infty} R(\lambda) KI_2(\lambda) = 0. \quad (5.15)$$

By substituting the Eq. (5.12) and Eq. (5.15) into Eq. (5.5) yields to

$$\lim_{\lambda \to \infty} \varphi_Y(\alpha) \equiv \varphi_0(\alpha) = \frac{\varphi_\zeta(\alpha) - 1}{i\alpha E(\zeta_1)}, \quad (5.16)$$

where $\varphi_\zeta(\alpha)$ is the characteristic function of the random variable ζ_1.

Proposition5.1. The limit function $\varphi_0(\alpha)$ is a characteristic function.

Proof. To prove this proposition, we construct a new distribution function as follows:

$$G(x) \equiv \frac{1}{E(\zeta_1)} \int_0^x \left(1 - \pi(z)\right) dz, \; x \geq 0. \quad (5.17)$$

Note that, G(x) is the limit distribution of a "residual waiting time" (Feller [6]). The characteristic function of $G(x)$ is equal to

$$\int_0^\infty e^{i\alpha x} dG(x) = \frac{1}{E(\zeta_1)} \int_0^\infty e^{i\alpha x} \left(1 - \pi(x)\right) dx = \frac{\varphi_\zeta(\alpha) - 1}{i\alpha E(\zeta_1)}. \quad (5.18)$$

Sowe see that $\varphi_0(\alpha) = \left(\varphi_\zeta(\alpha) - 1\right)/i\alpha E(\zeta_1)$ is the characteristic function of the limit distribution $G(x)$. This completes the proof of Proposition 5.1.

Theorem 5.2. Under the conditions of Theorem 3.1, the ergodic distribution $\left(Q_Y(x)\right)$ of the process $Y_\lambda(t)$ weakly converges to limit distribution $G(x)$, as $\lambda \to \infty$, that is,

$$\lim_{\lambda \to \infty} Q_Y(x) = G(x), \; x > 0$$

and the limit distribution function $G(x)$ has the following exact form:

$$G(x) \equiv \frac{1}{E(\zeta_1)} \int_0^x \left(1 - \pi(z)\right) dz. \quad (5.19)$$

$G(x)$ is the limit distribution for the "residualwaiting time" generated by the sequence $\{\zeta_n\}$.

Proof. By using Theorem 5.1, Proposition 5.1 and Continuity theorem for the characteristic functions, we can obtain the proof of Theorem 5.2. yields this theorem.

Taking account $\pi(z) = \frac{F(0) - F(-z)}{F(0)}$ into Eq. (5.19), we can obtain the following corollary.

Corollary 5.1. The following explicit form of the limit distribution $G(x)$ can be written:

$$G(x) = \frac{\int_0^x F_\eta(-z) dz}{\int_0^\infty F_\eta(-z) dz}, \quad x \geq 0. \quad (5.20)$$

In this section, two special cases are given.

Case 6.1. Let random variable η_1 have two-sided exponential distribution. In other words, the probability density function and distribution function of η_1 are defined as follows $(a > b > 0)$:

$$f_\eta(x) = \begin{cases} \frac{ab}{a+b} e^{ax}, & x < 0 \\ \frac{ab}{a+b} e^{-bx}, & x \geq 0 \end{cases} \quad ; \quad F_\eta(x) = \begin{cases} \frac{b}{a+b} e^{ax}, & x < 0 \\ 1 - \frac{a}{a+b} e^{-bx}, & x \geq 0 \end{cases}. \quad (6.1)$$

It immediately follows from Eq. (6.1) that,

$$F_\eta(-z) = \frac{b}{a+b} e^{-az}, \quad x \geq 0. \quad (6.2)$$

Hence, for each $x \geq 0$,

$$\int_0^x F_\eta(-z) dz = \frac{b}{a(a+b)} \left(1 - e^{-ax}\right) \quad \text{and} \quad \int_0^\infty F_\eta(-z) dz = \frac{b}{a(a+b)}. \quad (6.3)$$

Therefore, the explicit form of limit distribution (G(x)) can be derived from Eq. (5.20) by using Eq. (6.3)as follows:

$$G(x) = 1 - e^{-ax}, \; x \geq 0.$$

Case 6.2. Let η_1 be a random variable having uniform distribution over the interval $[-a, b]$, $b > a > 0$. In other words, assume that the probability density function and distribution function of η_1 are defined as follows:

$$f_\eta(x) = \begin{cases} \dfrac{1}{a+b}, & -a \le x \le b \\ 0, & \text{o.w.} \end{cases} \quad \text{and} \quad F_\eta(x) = \begin{cases} 0, & x < -a \\ \dfrac{x+a}{a+b}, & -a \le x \le b \\ 1, & x > b. \end{cases}$$

Because of that $F_\eta(-z) = \begin{cases} \frac{a-z}{a+b}, & z \le a \\ 0, & z > a \end{cases}$ for each $z \ge 0$, the following equalities can be written:

$$\int_0^\infty F_\eta(-z)\,dz = \frac{a^2}{2(a+b)} \quad \text{and} \quad \int_0^x F_\eta(-z)\,dz = \begin{cases} \dfrac{x^2}{2(a+b)}, & 0 \le x \le a \\ \dfrac{a^2}{2(a+b)}, & x > 0. \end{cases} \tag{6.4}$$

Hence, for each $x \ge 0$,

$$\int_0^x F_\eta(-z)\,dz = \frac{b}{a(a+b)}(1 - e^{-ax}); \quad \int_0^\infty F_\eta(-z)\,dz = \frac{b}{a(a+b)}. \tag{6.5}$$

Therefore, in this case, the explicit form of limit distribution (G(x)) can be obtained from Eq.(5.20) by using Eq. (6.5) as follows:

$G(x) = \min(1,\ x^2/a^2);\ x \ge 0.$

DISCUSSION

In this study, a semi – Markovian random walk with a generalized delaying barrier is considered and the ergodicity of this process is proved. Then, the characteristic function of the ergodic distribution of the process $X(t)$ is expressed by characteristic function of a boundary functional $S_{N(z)}$. Using representation (3.2), it is shown that the ergodic distribution of the "standardized" process $Y_\lambda(t) = X(t)/\lambda$ converges to limit distribution $(G(x))$ of a "residual waiting time", when $\lambda \to \infty$. Moreover, the explicit form of the limit distribution $G(x)$ is given by Eq. (5.20). Recall that limit distribution of "residual waiting time" is often used for solving various problems of stock control, queuing, reliability, mathematical insurance, stochastic finance, etc. Therefore, it is important to see that the ergodic distribution of the random walk with a generalized delaying barrier can be expressed (at least approximately!) by means of limit distribution of a "residual waiting time". Finally, in this study two special cases are considered and in both cases, the explicit forms of the limit distributions are established.

Note that, the asymptotic approach method considered here can be also used for obtaining approximation formulas which are simple enough for the ergodic distribution of the random walk process with other typesof barrier (e.g., reflecting, elastic, absorbing, etc.).

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