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Numerical method for Solving Fractional Advection-diffusion-wave Model

Rasool Kabiri, Reza chaharpashlou[1]
Department of Mathematics, Aliabad Katoul Branch
Islamic Azad University, Aliabad Katoul, Iran
University of technology Jundi-shapur, Department of basic science, Dezful, Iran
[1]Email: Chaharpashlou @jsu.ac.ir\

ABSTRACT
Fractional derivative operator is a useful theoretical discipline that is used by some of mathematicians for modeling and simulating many systems and processors based on the description of their properties.The main advantage of fractional derivatives in comparison with classical models with integer order is modeling many mechanical and electrical phenomena.One of these phenomena is the time fractional advection-diffusion-wave equation model (TFADE).In this paper this modelwith damping with index 1 <γ< 2 is considered. By usingthe implicit numerical method, TFADE model convergenceis achieved considerably, That finally two numerical examples are expressed.
***Keywords**: "Fractional differential equation, Fractional advection diffusion-wave, Implicit numerical method, Stability, Convergence".*

INTRODUCTION

In many studies of diffusion processes where the diffusion takes place in a highly non-homogeneous medium, numerous numerical experiments indicate that anomalous dispersion can not be described by the traditional second-order advection-dispersion equation (ADE) without extremely detailed information on the connectivity of high and low hydraulic conductivity sediments, it may even be that the ADE is insufficient at any level of detail [1]. For showing the details the differential equations with fractional derivatives can be used. Fractional-order differential equations has numerous applications in various sciences such as electronic. The fractional advection-dispersion equations have been recently treated by many authors. It is presented as a useful approach for the description of transport electronics in complex systems which are governed by anomalous diffusion and non-exponential relaxation patterns [2]. In this paper, the TFADE model with damping with index 1 <γ< 2 is considered. The effective implicit numerical method for this equation and its stability is discussed. Finally, two numerical examples are given.

TFADE Model with Index 1 <γ< 2

The time fractional advection-diffusion-wave equation with damping with index 1 < γ < 2 can be written as the following form [3].

$$\beta_2 \frac{\partial^\gamma c(x,t)}{\partial t^\gamma} + \beta_1 \frac{\partial c(x,t)}{\partial t} = -V \frac{\partial c(x,t)}{\partial x} + D \frac{\partial^2 c(x,t)}{\partial^2 x} + f(x,t). \qquad (1)$$

with boundary conditions:

$$c(a,t) = \phi_1(t), c(b,t) = \phi_2(t), \; 0 \le t \le T.$$

and initial conditions:

$$c(x,0) = \varphi_0(x), \; \frac{\partial c}{\partial t} = \varphi_1(x), a \le x \le b.$$

The parameters $\beta_1 \ge 0$, $\beta_2 \ge 0$, V>0, D>0 are known parameters; V is the drift of the process, that is, the mean adjective velocity, D is the coefficient of dispersion. This partial differential equation with γ = 2 and V = 0 is called the telegraph equation which governs electrical transmission in a telegraph cable. It can also be Characterized as a fractional diffusion-wave equation (governs wave motion in a string) with a damping effect due to the terms 1 <γ<2,V=0, $\beta_1 \frac{\partial c(x,t)}{\partial t}$ in the Eq (1). Here we can see there is some initial directionality to the wave motion, but this rapidly disappears and the motion becomes completely random. If 1<γ< 2, V = 0, $\beta_1 = 0,$ Eq (1) reduces to the fractional diffusion-wave equation. The most frequently encountered definition of fractional derivatives is based on Caputo derivatives as follow, [4].

Definition. For functions c(x,t) given in the interval [0,T], the expressions:

$$D_t^\gamma c(x,t) = \begin{cases} \frac{1}{\Gamma(m-\gamma)} \int_0^t \frac{c^m(\eta)}{(t-\eta)^{1+\gamma-m}} d\eta, \to m-1<\gamma<m \\ \frac{d^m}{dt^m} f(t), \to \gamma = m \in N. \end{cases} \quad (2)$$

is called time Caputo fractional derivative of order γ (m-1<γ<m).

Implicit Numerical Method for the TFADE

In this section, we propose an implicit numerical method, which can be used to solve the TFADE. Now we construct implicit numerical method using a new solution technique.

We define $t_k = k\tau$, k=0,1,...,n; $x_i = a + ih$, i=0,1,...,m; where $\tau = \frac{T}{n}$ and $h = \frac{(b-a)}{m}$ are space and time step sizes, respectively. Assume that $c(x,t) \in c^2([a,b] \times [0,T])$.

the Caputo time fractional derivative is discretized by using the following equation [5-6]:

$$\frac{\partial^\gamma c(x,t_{k+1})}{\partial t^\gamma} = \frac{\tau^{-\gamma}}{\Gamma(2-\lambda)} \sum_{j=0}^{k} b_j^\gamma [c(x,t_{k+1-j}) - c(x,t_{k-j}) + o(\tau^{2-\gamma}) \quad (3)$$

Where $b_j^\gamma = (j+1)^{1-\gamma} - j^{1-\gamma}$, j=0,1,2,...,n.

The first order spatial derivative can be approximated by the backward difference scheme:

$$\frac{\partial c(x_i,t_{k+1})}{\partial x} = \frac{c(x_i,t_{k+1}) - c(x_{i-1},t_{k+1})}{h} + o(h). \quad (4)$$

The first order temporal derivative can be approximated by the backward difference scheme:

$$\frac{\partial c(x_i,t_{k+1})}{\partial t} = \frac{c(x_i,t_{k+1}) - c(x_i,t_k)}{\tau} + o(\tau). \quad (5)$$

Lemma 1. If $0 < \bar{\gamma} = \gamma - 1 < 1$, there is a positive constant ρ such that:

$$\forall \text{ k=1,2,...}; (b_k^{\bar{\gamma}})^{-1} \le \rho k^{\bar{\gamma}}. \quad (6)$$

Proof [4]:

$$\lim_{k\to\infty} \frac{b_k^{-1}}{k^{\bar{\gamma}}} = \lim_{k\to\infty} \frac{k^{-\bar{\gamma}}}{(k+1)^{1-\bar{\gamma}} - k^{1-\bar{\gamma}}} = \lim_{k\to\infty} \frac{k^{-1}}{(1+\frac{1}{k})^{1-\bar{\gamma}} - 1} = \lim_{k\to\infty} \frac{k^{-1}}{(1-\bar{\gamma})^{k-1}} = \frac{1}{1-\lambda}.$$

By choosing $\rho = \frac{1}{1-\bar{\gamma}}$, Lemma is proved. ■

Let $U(x,t)=\frac{\partial c(x,t)}{\partial t}$, the Eq (1) can be rewritten as:

$$\begin{cases} \frac{\partial c}{\partial t}=U \\ \beta_2\frac{\partial^{\bar{\gamma}}U}{\partial t^{\bar{\gamma}}}+\beta_1 U=-V\frac{\partial c(x,t)}{\partial x}+D\frac{\partial^2 c(x,t)}{\partial x^2}+f(x,t) \end{cases} \quad (7)$$

where $0<\bar{\gamma}<1$. Hence, we can obtain the following difference scheme:

$$\begin{cases} \frac{\Delta_t c_i^k}{\tau}=U_i^{k+1} \\ \beta_2\frac{\tau^{-\bar{\gamma}}}{\Gamma(2-\bar{\gamma})}\sum_{j=0}^{k}b_j^{\bar{\gamma}}\Delta_t U_i^{k-j}+\beta_1 U_i^{k+1}=-V\frac{c_i^{k+1}-c_{i-1}^{k+1}}{h}+D\frac{c_{i-1}^{k+1}-2c_i^{k+1}+c_{i-1}^{k+1}}{h^2}+f_i^{k+1}, \end{cases} \quad (8)$$

where $\Delta_t c_i^k=c_i^{k+1}-c_i^k$ and $f_i^{k+1}=f(x_i,t_{k+1})$. Thus, the above second equation can be rewritten as:

$$(\beta_2+\beta_1\mu+r_1+2r_2)c_i^{k+1}-(r_1+r_2)c_{i-1}^{k+1}-r_2c_{i+1}^{k+1}=(\beta_2+\beta_1\mu)c_i^k+\tau[b_k^{\bar{\gamma}}U_i^0+\sum_{j=0}^{k-1}(b_j^{\bar{\gamma}}-b_{j+1}^{\bar{\gamma}})U_i^{k-j}]+\mu\tau f_i^{k+1}.$$

(9)

where $\mu=\tau^{\bar{\gamma}}\Gamma(2-\bar{\gamma})$, $r_1=V\,{}^{\mu\tau}\!/_{h}$, $r_2=D\,{}^{\mu\tau}\!/_{h^2}$.

Stability of the Implicit Numerical method for the TFADE

In this section, we discuss the implicit numerical methods (9) with boundary conditions $c_0^k=c_m^k=0$, k=0,1,2,…,n.

Definition. A finite difference scheme is said to be stable for the norm $\|.\|$, if there exists two constants $\gamma_1>0$ and $\gamma_2>0$, independent of h and τ, such that when h and τ tend towards zero then:

$$\|c^k\|\le\gamma_1\|c^0\|+\gamma_2\|f\|,\ \forall k>0.$$

whatever the initial data c^0 and the source term f .Let $\|c^{k+1}\|_\infty=\max_{1\le i\le m-1}\left|c_i^{k+1}\right|$ and $\|f\|_\infty=\max_{0\le i\le m;0\le k\le n.}\left|f_i^k\right|$.

Lemma 2. Assume that $c_i^k\ (i=1,2,\ldots,m-1;k=1,2,\ldots,n)$ be the numerical solution of (9). If $\left|c_{i_0}^{k+1}\right|=\max_{1\le i\le m-1}\left|c_i^{k+1}\right|$, then

$$\left[(\beta_2+\beta_1\mu)+\tau\right]\left\|c_i^{k+1}\right\|_\infty\le\left|(\beta_2+\beta_1\mu)c_{i_0}^k+\tau\left[b_k^{\bar{\gamma}}c_{i_0}^k+\sum_{j=0}^{k-1}\left(b_j^{\bar{\gamma}}-b_{j-1}^{\bar{\gamma}}\right)c_{i_0}^{k-j}\right]+\mu\tau f_i^{k+1}\right|.$$

Proof.

$$\left[(\beta_2+\beta_1\mu)+\tau\right]\left\|c^{k+1}\right\|_\infty=\left|\left[(\beta_2+\beta_1\mu)+\tau\right]c_{i_0}^{k+1}\right|\le(\beta_2+\beta_1\mu+r_1+2r_2)\left|c_{i_0}^{k+1}\right|-(r_1+r_2)\left|c_{i_0-1}^{k+1}\right|-r_2\left|c_{i_0+1}^{k+1}\right|$$

$$\le\left|(\beta_2+\beta_1\mu+r_1+2r_2)c_{i_0}^{k+1}-(r_1+r_2)c_{i_0-1}^{k+1}-r_2c_{i_0+1}^{k+1}\right|$$

$$\le\left|(\beta_2+\beta_1\mu)c_{i_0}^k+\tau[b_k^{\bar{\gamma}}c_{i_0}^0+\sum_{j=0}^{k-1}(b_j^{\bar{\gamma}}-b_{j+1}^{\bar{\gamma}})c_{i_0}^{k-j}]+\mu\tau f_i^{k+1}\right|.$$

Therefore, the conclusion of the Lemma 2 is proved. ■

Theorem 1. Assume that $c_i^k\ (i=1,2,\ldots,m-1;k=1,2,\ldots,n)$ be the numerical solution of (9). If $\beta_1\ne0$, then

$$(\beta_2+\beta_1\mu+\tau)\max_{1\le j\le k}\left\|c^j\right\|_\infty\le(\beta_2+\beta_1\mu+\tau)\left\|c^0\right\|_\infty+k\,\mu\tau\|f\|_\infty.$$

Further, we have

$$\left\|c^{k}\right\|_{\infty} \le \left\|c^{0}\right\|_{\infty} + \frac{k\,\mu\tau}{\beta_1}\left\|f\right\|_{\infty.}$$

Proof. Let $\left|c_{i_0}^{k+1}\right| = \max_{1\le i\le m-1}\left|c_{i}^{k+1}\right|$. by Lemma 2, we obtain:

$$\left|[(\beta_2+\beta_1\mu)+\tau]c_{i_0}^{k+1}\right| \le \left|(\beta_2+\beta_1\mu)c_{i_0}^{k} + \tau\left[b_k^{\bar\gamma}c_{i_0}^{k} + \sum_{j=0}^{k-1}\left(b_j^{\bar\gamma} - b_{j-1}^{\bar\gamma}\right)c_{i_0}^{k-j}\right] + \mu\tau f_i^{\,k+1}\right|$$

$$\le \left\{(\beta_2+\beta_1\mu)+\tau\left[b_k^{\bar\gamma}c_{i_0}^{k} + \sum_{j=0}^{k-1}\left(b_j^{\bar\gamma} - b_{j-1}^{\bar\gamma}\right)\right]\right\}\max_{1\le j\le k}\left\|c^{j}\right\|_{\infty} + \mu\tau\left\|f\right\|_{\infty}$$

$$= \left\{(\beta_2+\beta_1\mu)+\tau\right\}\max_{1\le j\le k}\left\|c^{j}\right\|_{\infty} + \mu\tau\left\|f\right\|_{\infty}$$

i.e.,

$$[(\beta_2+\beta_1\mu)+\tau]\max_{1\le j\le k+1}\left\|c^{j}\right\|_{\infty} \le [(\beta_2+\beta_1\mu)+\tau]\max_{1\le j\le k}\left\|c^{j}\right\|_{\infty} + \mu\tau\left\|f\right\|_{\infty}.$$

Thus,

$$[(\beta_2+\beta_1\mu)+\tau]\max_{1\le j\le k+1}\left\|c^{j}\right\|_{\infty} \le [(\beta_2+\beta_1\mu)+\tau]\left\|c^{0}\right\|_{\infty} + k\,\mu\tau\left\|f\right\|_{\infty}.$$

Therefore, the conclusion of the theorem is proved. ■

Theorem2. Assume that $c_i^k\ (i=1,2,\ldots,m-1;k=1,2,\ldots,n)$ be the numerical solution of (9). If $\beta_1 = 0$ and $\beta_2 \ne 0$, then:

$$\left\|c^{j}\right\|_{\infty} \le \left\|c^{0}\right\|_{\infty} + \left[b_{j-1}^{\bar\gamma} + (\beta_2+\tau)\right]^{-1}\mu\tau\left\|f\right\|_{\infty},\ j=1,2,\ldots,n.$$

Further

$$\left\|c^{j}\right\|_{\infty} \le \left\|c^{0}\right\|_{\infty} + \lambda\left(j\,\mu\tau\right)^{\bar\gamma}\left\|f\right\|_{\infty},\ j=1,2,\ldots,n.$$

Where λ is a positive constant.

Proof. For j=1, we assume that $\left|c_{i_0}^{1}\right| = \max_{1\le i\le m-1}\left|c_{j}^{1}\right|$. applying Lemma 2, we have

$$\left|(\beta_2+\tau)c_{i_0}^{1}\right| \le \left|(\beta_2+\tau)b_0^{\bar\gamma}c_{i_0}^{1} + \mu\tau f_{i_0}^{1}\right|.$$

Thus, $(\beta_2+\tau)\left\|c^{1}\right\|_{\infty} \le (\beta_2+\tau)\left\|c^{0}\right\|_{\infty} + \mu\tau\left\|f\right\|_{\infty}$, i.e.,

$$\left\|c^{1}\right\|_{\infty} \le \left\|c^{0}\right\|_{\infty} + \left[b_0^{\bar\gamma}(\beta_2+\tau)\right]^{-1}\mu\tau\left\|f\right\|_{\infty}.$$

Suppose that

$$\left\|c^{j}\right\|_{\infty} \le \left\|c^{0}\right\|_{\infty} + \left[b_{j-1}^{\bar\gamma}(\beta_2+\tau)\right]^{-1}\mu\tau\left\|f\right\|_{\infty},\ 1\le j\le k. \qquad (10)$$

When j=k+1, we assume that $\left|c_{i_0}^{k+1}\right| = \max_{1\le i\le m-1}\left|c_{j}^{k+1}\right|$. by Lemma 2, we obtain

$$\left|(\beta_2+\tau)c_{i_0}^{k+1}\right| \le \left|(\beta_2+\tau)\left[b_k^{\bar\gamma}c_{i_0}^{0} + \sum_{j=0}^{k-1}\left(b_j^{\bar\gamma} - b_{j+1}^{\bar\gamma}\right)c_{i_0}^{k-j}\right] + \mu\tau f_i^{\,k+1}\right|.$$

Therefore,

$$(\beta_2+\tau)\left\|c^{k+1}\right\| \le (\beta_2+\tau)\left[b_k^{\bar\gamma}\left\|c^{0}\right\| + \sum_{j=0}^{k-1}\left(b_j^{\bar\gamma} - b_{j+1}^{\bar\gamma}\right)\left\|c^{k-j}\right\|\right] + \mu\tau f_i^{\,k+1}.$$

Using the induction hypothesis (10) and $\left(b_j^{\bar\gamma}\right)^{-1} < \left(b_k^{\bar\gamma}\right)^{-1}\ (0\le j\le k-1)$, we obtain

$$\left\|c^{j}\right\|_{\infty} \leq \left\|c^{0}\right\|_{\infty} + \left[b_{k}^{\bar{\gamma}}\left(\beta_{2}+\tau\right)\right]^{-1} \mu\tau \left\|f\right\|_{\infty}, 1 \leq j \leq k.$$

Thus, we have

$$\left(\beta_{2}+\tau\right)\left\|c^{k+1}\right\| \leq \left(\beta_{2}+\tau\right)\left[b_{k}^{\bar{\gamma}} + \sum_{j=0}^{k-1}\left(b_{j}^{\bar{\gamma}} - b_{j+1}^{\bar{\gamma}}\right)\right]\left\|c^{0}\right\|$$

$$+\left(\beta_{2}+\tau\right)\sum_{j=0}^{k-1}\left(b_{j}^{\bar{\gamma}} - b_{j+1}^{\bar{\gamma}}\right)\left[b_{k}^{\bar{\gamma}}\left(\beta_{2}+\tau\right)\right]^{-1} \mu\tau\left\|f\right\|_{\infty} + \mu\tau\left\|f\right\|_{\infty}.$$

Further, we have

$$\left\|c^{k+1}\right\| \leq \left[b_{k}^{\bar{\gamma}} + \sum_{j=0}^{k-1}\left(b_{j}^{\bar{\gamma}} - b_{j+1}^{\bar{\gamma}}\right)\right]\left\|c^{0}\right\| + \sum_{j=0}^{k-1}\left(b_{j}^{\bar{\gamma}} - b_{j+1}^{\bar{\gamma}}\right)\left[b_{k}^{\bar{\gamma}}\left(\beta_{2}+\tau\right)\right]^{-1} \mu\tau\left\|f\right\|_{\infty} + \left(\beta_{2}+\tau\right)^{-1}\mu\tau\left\|f\right\|_{\infty}.$$

i.e.,

$$\left\|c^{k+1}\right\|_{\infty} \leq \left\|c^{0}\right\|_{\infty}\left[b_{k}^{\bar{\gamma}} + \sum_{j=0}^{k-1}\left(b_{j}^{\bar{\gamma}} - b_{j+1}^{\bar{\gamma}}\right)\right] + \left[b_{k}^{\bar{\gamma}}\left(\beta_{2}+\tau\right)\right]^{-1}\mu\tau\left\|f\right\|_{\infty} = \left\|c^{0}\right\|_{\infty} + \left[b_{k}^{\bar{\gamma}}\left(\beta_{2}+\tau\right)\right]^{-1}\mu\tau\left\|f\right\|_{\infty}.$$

Therefore

$$\left\|c^{j}\right\|_{\infty} \leq \left\|c^{0}\right\|_{\infty} + \left[b_{j-1}^{\bar{\gamma}} + (\beta_{2}+\tau)\right]^{-1}\mu\tau\left\|f\right\|_{\infty},\ j = 1,2,\ldots,n.$$

By Lemma 1, we obtain

$$\left\|c^{j}\right\|_{\infty} \leq \left\|c^{0}\right\|_{\infty} + \lambda\left(j\,\mu\tau\right)^{\bar{\gamma}}\left\|f\right\|_{\infty},\ j = 1,2,\ldots,n.$$

Where $\lambda = \rho\left(\beta_{2}+\tau\right)^{-1}\Gamma(2-\gamma)$.

Hence, the conclusion of the theorem is proved. ■

Theorem 3. The fractional implicit numerical method defined by (9) is unconditionally stable.

Proof. Let $\tilde{c}_{i}^{k}, (0 \leq i \leq m; 0 \leq j \leq n)$ be the approximate solution of (9), the error $\varepsilon_{i}^{k} = \tilde{c}_{i}^{k} - c_{i}^{k}, \left(0 \leq i \leq m; 0 \leq k \leq n\right)$ satisfies

$$(\beta_{2}+\beta_{1}\mu+r_{1}+2r_{2})\varepsilon_{i}^{k+1} - (r_{1}+r_{2})\varepsilon_{i-1}^{k+1} - r_{2}\varepsilon_{i+1}^{k+1} = (\beta_{2}+\beta_{1}\mu)\varepsilon_{i}^{k} + \tau[b_{k}^{\bar{\gamma}}\varepsilon_{i}^{0} + \sum_{j=0}^{k-1}(b_{j}^{\bar{\gamma}} - b_{j+1}^{\bar{\gamma}})\varepsilon_{i}^{k-j}] + \mu\tau f_{i}^{k+1}.$$

Applying theorems 2 and 3, we can obtain

$\left\|E^{k}\right\|_{\infty} \leq \left\|E^{0}\right\|_{\infty}$, k=1,2,...,n.

Where $\left\|E^{k}\right\|_{\infty} = \max_{1 \leq i \leq m-1}\left|\varepsilon_{i}^{k}\right|$. the conclusion of the theorem is proved. ■

Convergens of the Implicit Numerical method for the TFADE

Now we investigate the convergence of the implicit numerical methods (9).

Let $c(x_{i}, t_{k}), \left(i = 1,2,\ldots,m-1; k = 1,2,\ldots,n\right)$ be the exact solution of the Equation (1), At mesh point (x_{i}, t_{k}). define $\eta_{i}^{k} = c(x_{i}, t_{k}) - c_{i}^{k}$, i=1,2,...,m-1; k=1,2,...,n and $Y^{k} = (\eta_{1}^{k}, \eta_{2}^{k}, \ldots, \eta_{m-1}^{k})^{T}$. from (7) and (8), we obtain:

$$(\beta_{2}+\beta_{1}\mu+r_{1}+2r_{2})\eta_{i}^{k+1} - (r_{1}+r_{2})\eta_{i-1}^{k+1} - r_{2}\eta_{i+1}^{k+1} = (\beta_{2}+\beta_{1}\mu)\eta_{i}^{k} + \tau[b_{k}^{\bar{\gamma}}\eta_{i}^{0} + \sum_{j=0}^{k-1}(b_{j}^{\bar{\gamma}} - b_{j+1}^{\bar{\gamma}})\eta_{i}^{k-j}] + \tau R(x_{i}, t_{k}).$$

Theorem 4. Let c_{i}^{k} be the numerical solution computed by use of the implicit numerical methods (9), c(x,t) is the
solution of the problem (1). Then there is a positive constant A, such that

$$\left|c_i^k - c(x_i, t_k)\right| \le \begin{cases} A(\tau + h), \rightarrow when\, \beta_1 \ne 0, \\ A(\tau^{2-\gamma} + h), \rightarrow when\, \beta_1 = 0, \beta_2 \ne 0, \end{cases}$$

Where i=1,2,...,m-1; k=1,2,...,n.

Proof. Applying theorems 1 and 2 the conclusion of the theorem is proved. ∎

Numerical Results

In order to demonstrate the effectiveness of our theoretical analysis, two examples is now presented.

Example 1. Consider the following advection-diffusion-wave model with damping with index $1 < \gamma < 2$. the initial and boundary conditions are given by:

$$c(x,0) = 0,\ c_t(x,0) = 0,\ 0 < x < 1.$$

$$c(0,t) = t^3,\ c(1,t) = t^3 e,\ 0 \le t \le 1.$$

$$\frac{\partial^\gamma c(x,t)}{\partial t^\gamma} = -\frac{\partial c(x,t)}{\partial t} + \left(\frac{1}{2} + \frac{q}{2}\right)\frac{\partial^\alpha c(x,t)}{\partial x^\alpha} + D\left(\frac{1}{2} - \frac{q}{2}\right)\frac{\partial^\alpha c(x,t)}{\partial (-x)^\alpha} + f(x,t). \quad (11)$$

We take $f(x,t) = 3[t^2 + 2t^{2-\gamma} / \Gamma(3-\gamma)]e^x$.

where γ = 1.2 The exact solution of the equation is $u(x,t) = t^3 e^x$.

Comparison between exact and numerical solution is shown in figure 1. From Figure 1, it can be seen that the numerical solution is in excellent agreement with the exact solution.

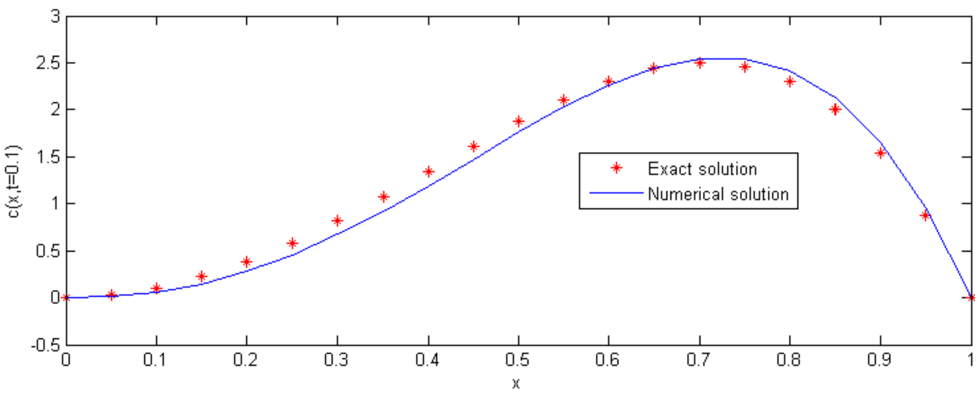

Figure (1) Comparison between the exact solution and the numerical solution when T=1 with $\gamma = 1.2$ in example 1.

Example 2. Consider the following time fractional advection–diffusion-wave model with damping with index $1 < \gamma < 2$.

with the initial and boundary conditions given by:

$$c(x,0) = \delta(x),\ \frac{\partial c(x,0)}{\partial t} = \delta(x),\ 0 \le x \le b.$$

$$c(0,t) = 0,\ \frac{\partial c(b,t)}{\partial x} = 0,\ 0 < t \le T.$$

$$\frac{\partial^\gamma c(x,t)}{\partial t^\gamma} + \frac{\partial c(x,t)}{\partial t} = -\frac{\partial c(x,t)}{\partial x} + \frac{\partial^2 c(x,t)}{\partial x^2}. \quad (12)$$

The computational results for different γ at T = 20 and x = 26 are shown in Figure 2 and Figure 3, respectively.

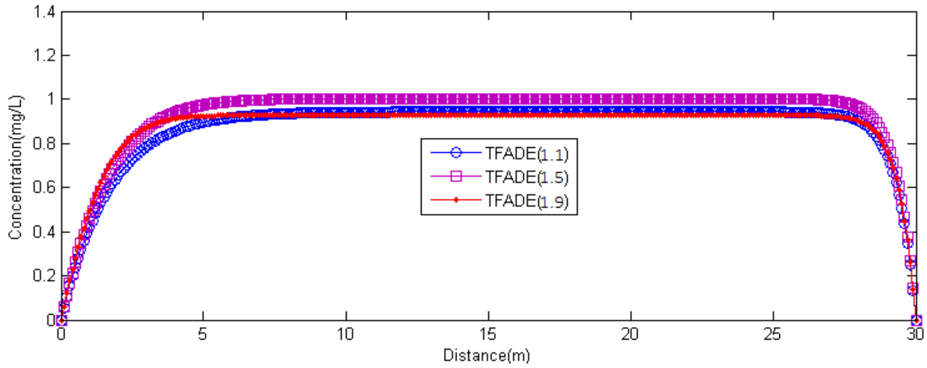

Figure (2) The snapshot of tracers whose transport is governed by the TFADE (11) when T=20 in example 2.

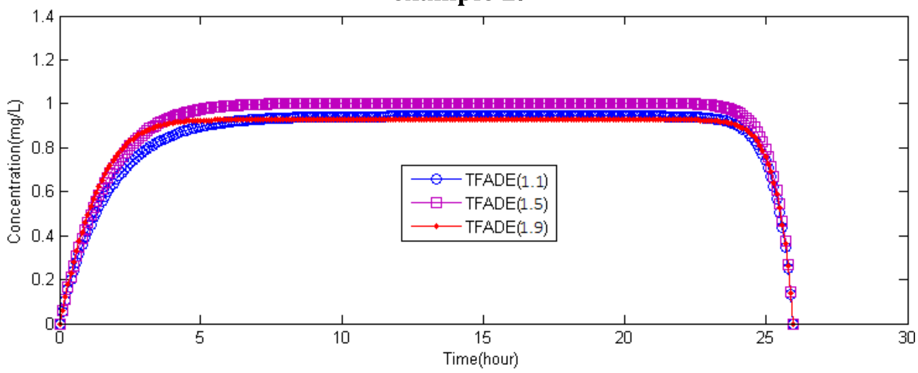

Figure (3) The snapshot of tracers whose transport is governed by the TFADE (12) when x=26 in example 2.

COCLUSIONS

In this paper, effective numerical method for solving the time fractional advection-diffusion-wave models has been described. Finally, result is given to demonstrate the effectiveness of theoretical analysis. This equation can be used to simulate the regional-scale anomalous dispersion with heavy tails. The method and technique discussed in this paper can also be applied to solve other kinds of fractional partial differential equations.

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